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A noncommutative geometric analysis of a sphere–torus topology change

J. Gratus*

Institute of Mathematics, University of Oslo, Akebergveien 56 leil 315, 0650 Olso, Norway

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Abstract

A one parameter set of noncommutative complex algebras is given. These may be considered deformation quantisation algebras. The commutative limit of these algebras correspond to the algebra of polynomial functions over a manifold or variety. The topology of the manifold or variety depends on the parameter, varying from nothing, to a point, a sphere, a certain variety and finally a torus. The irreducible adjoint preserving representations of the noncommutative algebras are studied. As well as typical noncommutative sphere type representations and noncommutative torus type representations, a new object is discovered and called a sphere–torus.

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1. Introduction

In noncommutative geometry we often wish to find analogues to topological properties of a manifold such as compactness, connectedness and genus, which we will consider here. For matrix geometries [1] the question of genus is tricky since both the sphere and torus have matrix analogues.

In deformation quantisation [2], one can simply take the commutative limit and ask what the genus of the underlying manifold is. In the case of the sphere and torus, we can also take the representation of the noncommutative algebra and compare its properties with the representations of the noncommutative sphere and torus.

* Tel.: +47-22-67-56-56; fax: +47-22-85-43-49.
E-mail address: j@gratus.net (J. Gratus).

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In this paper we present, in Section 2, a one parameter set of deformation algebras $\mathcal{A}(R)$ for $R \in \mathbb{R}$. The commutative limit of these algebras are $C_0^\omega(\mathcal{M}(R))$ the commutative algebra of complex polynomials on the manifold (or variety) $\mathcal{M}(R)$. This manifold has different topologies depending on the value of R. Varying from nothing, to a point, a sphere, a variety and finally a torus. This is described in Section 1.2.

In Section 3 we look at the finite dimensional representations of $\mathcal{A}(R)$. These can be classified as either S^2-type representations or T^2-type representations by comparing them to the representations for the noncommutative torus or the noncommutative sphere. Depending on the value of R one or other of these representations exists. What we show in this paper is that there is a region of R where both types of representation exist. This region, which we shall name the *sphere–torus*, is a purely noncommutative region, it disappears in the commutative limit.

We summarise the various representation in the conclusion, Section 4.

1.1. Notation

$\mathbb{R}_+$	$\{t \in \mathbb{R} \| t > 0\}$
$\mathcal{M}$	manifold or variety
$C^\omega(\mathcal{M})$, $C^\omega(\mathbb{R}^r)$	commutative algebra of complex analytic functions over $\mathcal{M}$ or $\mathbb{R}^r$
$C_0^\omega(\mathcal{M})$, $C_0^\omega(\mathbb{R}^r)$	commutative algebra of complex polynomials over $\mathcal{M}$ or $\mathbb{R}^r$
$\mathcal{A}$, $\mathcal{B}$	noncommutative algebras over $\mathbb{C}$
Unbold symbols: $f, g, u, v, x_i, F_s, w, x, y, z$	elements of the commutative algebras, i.e. analytic or polynomial functions over $\mathcal{M}$ or $\mathbb{R}$
Bold symbols: $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}_i, \boldsymbol{F}_s, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	elements of the noncommutative algebras $\mathcal{A}$ and $\mathcal{B}$
$\boldsymbol{\varepsilon}$	element in the centre of the noncommutative algebras $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{H}$	finite or infinite dimensional Hilbert space
$\mathcal{L}(\mathcal{H}, \mathcal{H})$	space of linear maps over $\mathcal{H}$ (bounded or unbounded operators)
$M_n(\mathbb{C})$	space of $n \times n$ complex matrices
$\mathbb{1}$	the identity in $\mathcal{L}(\mathcal{H}, \mathcal{H})$ and the unit matrix in $M_n(\mathbb{C})$
$\|\theta\rangle$	bra-ket notation for vectors in $\mathcal{H}$

1.2. A one parameter set of immersions with a sphere, torus and variety

Consider the immersions given by

$$\mathcal{M}(R) = \{(x, y, z) \in \mathbb{R}^3 | z^2 + (x^2 + y^2 - R)^2 = 1\}. \tag{1}$$

It is obvious that for $R < -1$ there is no solution to (1) while for $R = -1$, $\mathcal{M}(R)$ consists of the single point at the origin $(x, y, z) = (0, 0, 0)$. To picture $\mathcal{M}(R)$ for $R > -1$, we note that it is axisymmetric about the z-axis. Therefore we can examine the shape of $\mathcal{M}(R)$

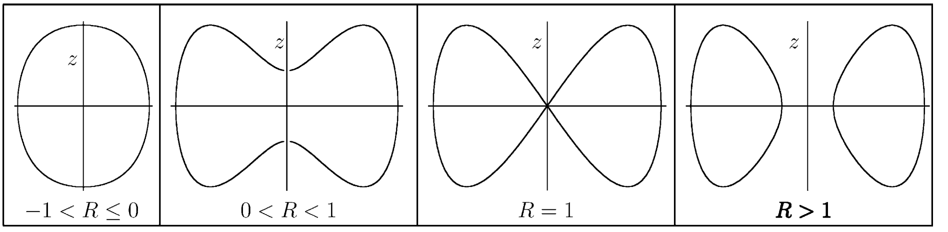

Fig. 1. Shape of the slice of $\mathcal{M}$ setting $y = 0$. The shape of $\mathcal{M}$ is given by rotating slice about the z-axis.

by setting $y = 0$ and rotating the subsequent one dimensional variety about the z-axis. From Fig. 1 we can see that for $-1 < R \leq 0$, $\mathcal{M}(R)$ is a convex manifold topologically equivalent to the sphere. For $0 < R < 1$, $\mathcal{M}(R)$ is a nonconvex manifold topologically equivalent to the sphere. For $R = 1$, $\mathcal{M}(R)$ is not a manifold but instead a 2D variety, which is smooth about all points except the origin $(x, y, z) = (0, 0, 0)$. For $R > 1$, $\mathcal{M}(R)$ is a torus.

There is a Poisson structure on $\mathcal{M}$ is given by

$$\{x, y\} = z, \qquad \{z, x\} = 2(x^2 + y^2 - R)y = 2wy,$$
$$\{y, z\} = 2(x^2 + y^2 - R)x = 2wx, \tag{2}$$

where $w = x^2 + y^2 - R$, which is consistent with (1).

We can give a Darboux coordinate system (p, q) such that

$$x = (R + \cos(2p))^{1/2} \cos(q), \qquad y = -(R + \cos(2p))^{1/2} \sin(q),$$
$$z = \sin(2p), \tag{3}$$

where $\{p, q\} = 1$. It is easy to see that (1) and (2) are satisfied. A necessary condition for these to be valid is $R + \cos(2p) > 0$. Thus for the torus $R > 1$ this is valid for all p. More specifically we can patch coordinate systems with $0 < q < 2\pi$ and $0 < p < \pi$. For the variety $R = 1$ we must exclude the point $p = (1/2)\pi$ which correspond to the point at the origin. For $-1 < R < 1$ we have $-(1/2)(\pi - \arccos(R)) < p < (1/2)(\pi - \arccos(R))$, and we must exclude the two points $(x, y, z) = (0, 0, \pm(1 - R^2)^{1/2})$.

1.3. *Brief introduction to deformation quantisation*

We limit ourselves in this paper to a the deformation quantisation of algebraic manifolds and varieties with algebraic Poisson structures. Let $\mathcal{M}$ smooth m dimensional Poisson manifold or variety given by

$$\mathcal{M} = \{(x_1, \dots, x_r) \in \mathbb{R}^r | F_s(x_1, \dots, x_r) = 0, s = 1, \dots, r - m\}, \tag{4}$$

where $F_s(x_1, \dots, x_m)$ are polynomials. Let the Poisson structure $\{\bullet, \bullet\}$ be given by $\{x_i, x_j\} = C_{ij}(x_1, \dots, x_m)$, where $C_{ij}(x_1, \dots, x_m)$ are also polynomials. Consistency implies $\{x_i, F_s\} = 0$. Let $C^\omega(\mathcal{M})$ be the algebra of complex analytic functions on $\mathcal{M}$ and let $C_0^\omega(\mathcal{M})$ be the subalgebra of polynomials in $(x_1, \dots, x_m)$, this is dense in $C^\omega(\mathcal{M})$.

Let $\mathcal{B}$ be the free noncommutative algebra generated by $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r, \boldsymbol{\varepsilon}\}$, and define the linear map $\pi_{\mathcal{B}}: \mathcal{B} \mapsto C_0^{\omega}(\mathbb{R}^r)$ where $C_0^{\omega}(\mathbb{R}^r)$ is the algebra of polynomials in $(x_1, \ldots, x_n)$, via

$$\begin{aligned}&\pi_{\mathcal{B}}(\boldsymbol{f}+\boldsymbol{g}) = \pi_{\mathcal{B}}(\boldsymbol{f}) + \pi_{\mathcal{B}}(\boldsymbol{g}), \qquad \pi_{\mathcal{B}}(\boldsymbol{fg}) = \pi_{\mathcal{B}}(\boldsymbol{f})\pi_{\mathcal{B}}(\boldsymbol{g}), \qquad \pi_{\mathcal{B}}(\lambda) = \lambda,\\ &\pi_{\mathcal{B}}(\boldsymbol{\varepsilon}) = 0, \qquad \pi_{\mathcal{B}}(\boldsymbol{x}_i) = x_i.\end{aligned} \tag{5}$$

Choose the elements $\boldsymbol{F}_s \in \mathcal{B}$ for $s = 1, \ldots, r-m$ and $\boldsymbol{C}_{ij} \in \mathcal{B}$ for $i, j = 1, \ldots, r$ such that $\pi_{\mathcal{B}}(\boldsymbol{F}_s) = F_s$ and $\pi_{\mathcal{B}}(\boldsymbol{C}_{ij}) = C_{ij}$. We define the algebra $\mathcal{A}$ to be $\mathcal{B}$ quotiented by the noncommutative polynomial relationships

$$[\boldsymbol{\varepsilon}, \boldsymbol{x}_i] = 0, \qquad \boldsymbol{F}_s = 0, \qquad [\boldsymbol{x}_i, \boldsymbol{x}_j] = \boldsymbol{\varepsilon}\boldsymbol{C}_{ij} \tag{6}$$

for $i, j = 1, \ldots, r$ and $s = 1, \ldots, r-m$. The first equation in (6) implies that $\boldsymbol{\varepsilon}$ is in the centre of $\mathcal{A}$. We demand that $\mathcal{A}$ be an associative algebra. This imposes restrictions on the possible choices of $\boldsymbol{F}_s$ and $\boldsymbol{C}_{ij}$, which we will not investigate here.

We can define the map

$$\pi: \mathcal{A} \mapsto C_0^{\omega}(\mathcal{M}), \quad \pi(\boldsymbol{\varepsilon}) = 0 \tag{7}$$

which is surjective. It is easy to see that this gives the Poisson structure via

$$\{\pi(\boldsymbol{f}), \pi(\boldsymbol{g})\} = \pi\left(\frac{1}{\mathrm{i}\boldsymbol{\varepsilon}}[\boldsymbol{f}, \boldsymbol{g}]\right). \tag{8}$$

Thus the following diagram commutes

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "/\{[\boldsymbol{x}_i,\varepsilon]=0,\ \boldsymbol{F}_s=0,\ [\boldsymbol{x}_i,\boldsymbol{x}_j]=\varepsilon\boldsymbol{C}_{ij}\}"] \arrow[d, "\pi_{\mathcal{B}}"'] & \mathcal{A} \arrow[d, "\pi"] \\
C_0^{\omega}(\mathbb{R}^r) \arrow[r, "/\{F_s=0\}"] & C_0^{\omega}(\mathcal{M})
\end{tikzcd} \tag{9}$$

We also demand that there is a conjugate structure $\dagger: \mathcal{A} \mapsto \mathcal{A}$ such that

$$(\boldsymbol{fg})^{\dagger} = \boldsymbol{g}^{\dagger}\boldsymbol{f}^{\dagger}, \qquad \lambda^{\dagger} = \bar{\lambda} \ \text{ for } \lambda \in \mathbb{C}, \qquad \pi(\boldsymbol{f}^{\dagger}) = \overline{\pi(\boldsymbol{f})}, \qquad \boldsymbol{\varepsilon}^{\dagger} = \boldsymbol{\varepsilon}. \tag{10}$$

We are interested in representations $\Psi: \mathcal{A} \mapsto \mathcal{L}(\mathcal{H}, \mathcal{H})$ where $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is the space of linear maps on the Hilbert space $\mathcal{H}$. We demand that Ψ is irreducible, and $\Psi(\boldsymbol{\varepsilon}) = \varepsilon\mathbb{1}$ where $\varepsilon \in \mathbb{R}_+$. If $\dim \mathcal{H} = n$ then $\mathcal{L}(\mathcal{H}, \mathcal{H}) \cong M_n(\mathbb{C})$. On the other hand if $\dim \mathcal{H} = \infty$ then $\mathcal{L}(\mathcal{H}, \mathcal{H})$ may contain unbounded operators. We also demand that Ψ preserve the conjugate structure $\Psi(\boldsymbol{f}^{\dagger}) = \Psi(\boldsymbol{f})^{\dagger}$ where the dagger on the right is the Hermitian conjugate or adjoint.

We use the bra-ket notation so that if $|\theta\rangle \in \mathcal{H}$ then $\Psi(\boldsymbol{f})|\theta\rangle$ the action of $\boldsymbol{f} \in \mathcal{A}$ on $|\theta\rangle$ is written $\boldsymbol{f}|\theta\rangle$. Since Ψ preserves the conjugate we have $\langle\theta'|\boldsymbol{f}^{\dagger}|\theta\rangle = \overline{\langle\theta|\boldsymbol{f}|\theta'\rangle}$.

We sometimes want to recover the commutative structure from the matrix algebras. This requires finding a sequence of representations $\Psi_n: \mathcal{A} \mapsto M_n(\mathbb{C})$, $\Psi_n(\boldsymbol{\varepsilon}) \to 0$ as $n \to \infty$. However in general there is no canonical map $M_n(\mathbb{C}) \mapsto M_{n+1}(\mathbb{C})$. One exception being the noncommutative sphere [3].

As stated above, in this paper we give a shall give a one parameter family of such algebras, whose representations can be compared to those for the sphere and torus. Here we give a brief summary of these two noncommutative geometries.

The *noncommutative sphere* is generated by $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{\varepsilon}$ with

$$[\boldsymbol{x},\boldsymbol{y}]=\mathrm{i}\boldsymbol{\varepsilon}\boldsymbol{z},\qquad [\boldsymbol{y},\boldsymbol{z}]=\mathrm{i}\boldsymbol{\varepsilon}\boldsymbol{x},\qquad [\boldsymbol{z},\boldsymbol{x}]=\mathrm{i}\boldsymbol{\varepsilon}\boldsymbol{y},\qquad \boldsymbol{x}^2+\boldsymbol{y}^2+\boldsymbol{z}^2=1,$$
$$\boldsymbol{x}^\dagger=\boldsymbol{x},\qquad \boldsymbol{y}^\dagger=\boldsymbol{y},\qquad \boldsymbol{z}^\dagger=\boldsymbol{z}. \tag{11}$$

The representations are the finite dimensional irreducible representations of SO(3). A basis for $\mathcal{H}$ is $|0\rangle,\dots,|n-1\rangle$ so that

$$\varepsilon=2(n^2-1)^{-1/2},\qquad \boldsymbol{a}_+|r\rangle=\varepsilon(n-r-1)^{1/2}(r+1)^{1/2}|r+1\rangle,$$
$$z|r\rangle=\varepsilon(r-\tfrac{1}{2}(n-1))|r\rangle,\qquad \boldsymbol{a}_-|r\rangle=\varepsilon(n-r)^{1/2}r^{1/2}|r-1\rangle, \tag{12}$$

where $\boldsymbol{a}_+=\boldsymbol{x}+\mathrm{i}\boldsymbol{y}$ and $\boldsymbol{a}_-=\boldsymbol{x}-\mathrm{i}\boldsymbol{y}$ are called the ladder elements. We note that the ladder operators, which are the representations of the ladder elements, terminate $\Psi_n(\boldsymbol{a}_+)^n=0$. We also note that Ψ is unique for each n, and that $\Psi_n(\boldsymbol{\varepsilon})\to 0$ as $n\to\infty$.

In our language we the *noncommutative torus* is generated by $\{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{\varepsilon}\}$ with

$$\boldsymbol{x}_1^2+\boldsymbol{x}_2^2=1,\qquad \boldsymbol{x}_3^2+\boldsymbol{x}_4^2=1,\qquad [\boldsymbol{x}_1,\boldsymbol{x}_2]=0,\qquad [\boldsymbol{x}_3,\boldsymbol{x}_4]=0,$$
$$[\boldsymbol{x}_1,\boldsymbol{x}_3]=-\mathrm{i}\boldsymbol{\varepsilon}(\boldsymbol{x}_2\boldsymbol{x}_4+\boldsymbol{x}_4\boldsymbol{x}_2),\qquad [\boldsymbol{x}_1,\boldsymbol{x}_4]=\mathrm{i}\boldsymbol{\varepsilon}(\boldsymbol{x}_2\boldsymbol{x}_3+\boldsymbol{x}_3\boldsymbol{x}_2),$$
$$[\boldsymbol{x}_2,\boldsymbol{x}_3]=\mathrm{i}\boldsymbol{\varepsilon}(\boldsymbol{x}_1\boldsymbol{x}_4+\boldsymbol{x}_4\boldsymbol{x}_1),\qquad [\boldsymbol{x}_2,\boldsymbol{x}_4]=-\mathrm{i}\boldsymbol{\varepsilon}(\boldsymbol{x}_1\boldsymbol{x}_3+\boldsymbol{x}_3\boldsymbol{x}_1),$$
$$\boldsymbol{x}_1^\dagger=\boldsymbol{x}_1,\qquad \boldsymbol{x}_2^\dagger=\boldsymbol{x}_2,\qquad \boldsymbol{x}_3^\dagger=\boldsymbol{x}_3,\qquad \boldsymbol{x}_4^\dagger=\boldsymbol{x}_4. \tag{13}$$

If we set $\boldsymbol{u}=\boldsymbol{x}_1+\mathrm{i}\boldsymbol{x}_2$, $\boldsymbol{v}=\boldsymbol{x}_3+\mathrm{i}\boldsymbol{x}_4$, and $\boldsymbol{q}=(1+\mathrm{i}\boldsymbol{\varepsilon})/(1-\mathrm{i}\boldsymbol{\varepsilon})$ then we derive the usual noncommutative torus.

$$\boldsymbol{u}\boldsymbol{u}^\dagger=\boldsymbol{u}^\dagger\boldsymbol{u}=1,\qquad \boldsymbol{v}\boldsymbol{v}^\dagger=\boldsymbol{v}^\dagger\boldsymbol{v}=1,\qquad \boldsymbol{u}\boldsymbol{v}=\boldsymbol{q}\boldsymbol{v}\boldsymbol{u}. \tag{14}$$

However as we have defined the algebra $\mathcal{A}$, the element $\boldsymbol{q}$ and $\boldsymbol{q}^\dagger$ are not members of $\mathcal{A}$. To solve this problem we say that $\mathcal{A}$ is generated by $\{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4,\boldsymbol{\varepsilon},(1+\boldsymbol{\varepsilon}^2)^{-1}\}$.

There are both finite and infinite representations of the noncommutative torus. The finite dimensional representations Ψ_n have a basis $|0\rangle,\dots,|n-1\rangle$

$$\Psi_n(\boldsymbol{q})=q\mathbb{1},\qquad q=\mathrm{e}^{2\pi\mathrm{i}k/n},\qquad \boldsymbol{u}|r\rangle=\mathrm{e}^{\mathrm{i}(\beta+2\pi rk\mathrm{i}/n)}|r\rangle,$$
$$\boldsymbol{v}|r\rangle=|r+1\rangle,\qquad \boldsymbol{v}|n-1\rangle=\nu|0\rangle, \tag{15}$$

where $n,k\in\mathbb{N}$ and where $\nu\in\mathbb{C}$, $|\nu|=1$ is a phase. We impose that n and k are relatively prime, so that there are no multiple eigenvalues of $\Psi(\boldsymbol{u})$. There also exist other more complicated finite dimensional representations of the noncommutative torus where $\Psi(\boldsymbol{u})$ has multiple eigenvalues.

We note that the ladder elements of this representation are given by $\boldsymbol{v}$ and $\boldsymbol{v}^\dagger$ and that the ladder operators do not terminate $\Psi_n(\boldsymbol{v})^m\neq 0$ for all $n,m\in\mathbb{Z}$. To specify Ψ_n completely requires giving (n,k,β,ν). If $k=1$ then $\Psi_n(\boldsymbol{q})\to 1$ as $n\to\infty$.

The infinite dimensional representations have a basis $|r\rangle$, $r \in \mathbb{Z}$, and are determined by the parameters $\alpha, \beta \in \mathbb{R}$ where $\alpha/2\pi$ is irrational

$$\Psi(\boldsymbol{q}) = q\mathbb{1}, \qquad q = \mathrm{e}^{\mathrm{i}\alpha}, \qquad \boldsymbol{u}|r\rangle = \mathrm{e}^{\mathrm{i}(r\alpha+\beta)}|r\rangle, \qquad \boldsymbol{v}|r\rangle = |r+1\rangle. \tag{16}$$

The eigenvalues of $\Psi(\boldsymbol{u})$ are dense on unit circle.

2. $\mathcal{A}(R)$: the deformation algebra of polynomials on $\mathcal{M}(R)$

In Section 1.2, we define a one parameter set of immersions $\mathcal{M}(R) \in \mathbb{R}^3$. The commutative algebra of complex valued polynomials in (x, y, z) on $\mathcal{M}(R)$ is written $C_0^{\omega}(\mathcal{M}(R))$ and of course is dense in $C^{\omega}(\mathcal{M}(R))$. Here we give a one parameter set of complex noncommutative algebras $\mathcal{A}(R)$ with $R \in \mathbb{R}$, which are the deformation quantisation of $C_0^{\omega}(\mathcal{M}(R))$. Each $\mathcal{A}(R)$ is generated by $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\varepsilon}, (1+\boldsymbol{\varepsilon}^2)^{-1}\}$ with $\boldsymbol{\varepsilon}$ in the centre of $\mathcal{A}(R)$. The reason for including $(1+\boldsymbol{\varepsilon}^2)^{-1}$ is similar as for the noncommutative torus. These are related via

$$[\boldsymbol{x}, \boldsymbol{y}] = \mathrm{i}\boldsymbol{\varepsilon}\boldsymbol{z}, \qquad [\boldsymbol{y}, \boldsymbol{z}] = \mathrm{i}\boldsymbol{\varepsilon}(\boldsymbol{w}\boldsymbol{x} + \boldsymbol{x}\boldsymbol{w}), \qquad [\boldsymbol{z}, \boldsymbol{x}] = \mathrm{i}\boldsymbol{\varepsilon}(\boldsymbol{w}\boldsymbol{y} + \boldsymbol{y}\boldsymbol{w}) \tag{17}$$

and

$$\boldsymbol{z}^2 + \boldsymbol{w}^2 = 1, \tag{18}$$

where we define the element $\boldsymbol{w} \in \mathcal{A}(R)$ via

$$\boldsymbol{w} = \boldsymbol{x}^2 + \boldsymbol{y}^2 - R. \tag{19}$$

The adjoint operation is given by

$$\boldsymbol{x}^{\dagger} = \boldsymbol{x}, \qquad \boldsymbol{y}^{\dagger} = \boldsymbol{y}, \qquad \boldsymbol{z}^{\dagger} = \boldsymbol{z}, \qquad \boldsymbol{w}^{\dagger} = \boldsymbol{w}. \tag{20}$$

It is easy to see that, assuming (17)–(19) define an associative algebra, then $\pi: \mathcal{A}(R) \mapsto C_0^{\omega}(\mathcal{M}(R))$, where $\pi(\boldsymbol{x}) = x$, $\pi(\boldsymbol{y}) = y$, $\pi(\boldsymbol{z}) = z$. It also gives with the correct Poisson structure. We shall show that these relationships define an associative algebra in Lemma 2. But first we need to define some new elements of $\mathcal{A}(R)$ and derive some relationships which are valid if $\mathcal{A}(R)$ is associative.

We define the ladder elements $\boldsymbol{a}_+, \boldsymbol{a}_- \in \mathcal{A}(R)$ via

$$\boldsymbol{a}_+ = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}, \qquad \boldsymbol{a}_- = \boldsymbol{x} - \mathrm{i}\boldsymbol{y}. \tag{21}$$

In order to emphasise the circular nature of $\boldsymbol{w}$ and $\boldsymbol{z}$ we shall define the element $\boldsymbol{u} \in \mathcal{A}(R)$ via

$$\boldsymbol{u} = \boldsymbol{w} + \mathrm{i}\boldsymbol{z} \in \mathcal{A}(R) \tag{22}$$

and we show below that $\boldsymbol{u}$ is unitary, i.e. $\boldsymbol{u}^{\dagger} = \boldsymbol{u}^{-1}$.

We define the "pseudo element" $\boldsymbol{\alpha}$ via

$$\boldsymbol{\varepsilon} = \tan\left(\tfrac{1}{2}\boldsymbol{\alpha}\right) \tag{23}$$

and observe that $\boldsymbol{\alpha}$ is not a member of $\mathcal{A}(R)$. However since $\boldsymbol{\varepsilon} \in \mathcal{A}(R)$ and $(1+\boldsymbol{\varepsilon}^2)^{-1} \in \mathcal{A}(R)$ then from the tan half angle formulae we have $\sin(\boldsymbol{\alpha}) = 2\boldsymbol{\varepsilon}(1+\boldsymbol{\varepsilon}^2)^{-1} \in \mathcal{A}(R)$ and

$\cos(\boldsymbol{\alpha}) = (1-\boldsymbol{\varepsilon}^2)(1+\boldsymbol{\varepsilon}^2)^{-1} \in \mathcal{A}(R)$. Also $\mathrm{e}^{\pm \mathrm{i}n\boldsymbol{\alpha}} = (\cos(\boldsymbol{\alpha}) \pm \mathrm{i}\sin(\boldsymbol{\alpha}))^n$ so $\mathrm{e}^{\mathrm{i}n\boldsymbol{\alpha}} \in \mathcal{A}(R)$ for all $n \in \mathbb{Z}$.

Before showing that the algebra $\mathcal{A}(R)$ is associative we derive some direct consequences of these definitions.

Lemma 1. *From the above definitions, and the assumption that $\mathcal{A}(R)$ is associative, we have the following relationships*:

$$[z, \boldsymbol{w}] = 0, \tag{24}$$

$$\boldsymbol{u}\boldsymbol{u}^\dagger = \boldsymbol{u}^\dagger\boldsymbol{u} = 1, \tag{25}$$

$$\boldsymbol{a}_+\boldsymbol{a}_- = \boldsymbol{w} + R + \boldsymbol{\varepsilon}z, \qquad \boldsymbol{a}_-\boldsymbol{a}_+ = \boldsymbol{w} + R - \boldsymbol{\varepsilon}z, \tag{26}$$

$$[\boldsymbol{w}, \boldsymbol{a}_+] = -\boldsymbol{\varepsilon}(z\boldsymbol{a}_+ + \boldsymbol{a}_+z), \qquad [\boldsymbol{w}, \boldsymbol{a}_-] = +\boldsymbol{\varepsilon}(z\boldsymbol{a}_- + \boldsymbol{a}_-z). \tag{27}$$

Also

$$\begin{aligned} z\boldsymbol{a}_+ &= \boldsymbol{a}_+(\cos(\boldsymbol{\alpha})z + \sin(\boldsymbol{\alpha})\boldsymbol{w}), \qquad \boldsymbol{w}\boldsymbol{a}_+ = \boldsymbol{a}_+(-\sin(\boldsymbol{\alpha})z + \cos(\boldsymbol{\alpha})\boldsymbol{w}),\\ z\boldsymbol{a}_- &= \boldsymbol{a}_-(\cos(\boldsymbol{\alpha})z - \sin(\boldsymbol{\alpha})\boldsymbol{w}), \qquad \boldsymbol{w}\boldsymbol{a}_- = \boldsymbol{a}_-(\sin(\boldsymbol{\alpha})z + \cos(\boldsymbol{\alpha})\boldsymbol{w}) \end{aligned} \tag{28}$$

or alternatively

$$\begin{aligned} &\boldsymbol{u}\boldsymbol{a}_+ = \boldsymbol{a}_+\boldsymbol{u}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}}, \qquad \boldsymbol{u}\boldsymbol{a}_- = \boldsymbol{a}_-\boldsymbol{u}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}}, \qquad \boldsymbol{u}^\dagger\boldsymbol{a}_+ = \boldsymbol{a}_+\boldsymbol{u}^\dagger\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}},\\ &\boldsymbol{u}^\dagger\boldsymbol{a}_+ = \boldsymbol{a}_-\boldsymbol{u}^\dagger\,\mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}}. \end{aligned} \tag{29}$$

The general element $\boldsymbol{f} \in \mathcal{A}(R)$ can be written uniquely in the form

$$\boldsymbol{f} = \sum_{r=0}^{N}\sum_{s=-N}^{N} \boldsymbol{a}_+^r\boldsymbol{u}^s\xi_{r,s}(\boldsymbol{\varepsilon}) + \sum_{r=1}^{N}\sum_{s=-N}^{N} \boldsymbol{a}_-^r\boldsymbol{u}^s\xi_{-r,s}(\boldsymbol{\varepsilon}) \tag{30}$$

for some $N \in \mathbb{N}$ where for all $r, s = -N, \dots, N$ the function $\xi_{r,s}(t)$ is a ration function in t with denominator $(1+t^2)^m$ for some $m \in \mathbb{N}$.

Proof. Eq. (26) follows automatically from (21), (19) and the first equation in (17). Eq. (27) follows from:

$$[\boldsymbol{w}, \boldsymbol{a}_+] = [\boldsymbol{a}_+\boldsymbol{a}_- - R - \boldsymbol{\varepsilon}z, \boldsymbol{a}_+] = \boldsymbol{a}_+[\boldsymbol{a}_-, \boldsymbol{a}_+] - \boldsymbol{\varepsilon}[z, \boldsymbol{a}_+] = -\boldsymbol{\varepsilon}(2\boldsymbol{a}_+z + [z, \boldsymbol{a}_+])$$

and likewise for $[\boldsymbol{w}, \boldsymbol{a}_-]$. For (24) we have

$$\begin{aligned} [z, \boldsymbol{w}] &= [z, \boldsymbol{a}_+\boldsymbol{a}_- - R - \boldsymbol{\varepsilon}z] = [z, \boldsymbol{a}_+]\boldsymbol{a}_- + \boldsymbol{a}_+[z, \boldsymbol{a}_-]\\ &= \boldsymbol{\varepsilon}(\boldsymbol{w}\boldsymbol{a}_+\boldsymbol{a}_- + \boldsymbol{a}_+\boldsymbol{w}\boldsymbol{a}_- - \boldsymbol{a}_+\boldsymbol{w}\boldsymbol{a}_- - \boldsymbol{a}_+\boldsymbol{a}_-\boldsymbol{w}) = \boldsymbol{\varepsilon}(\boldsymbol{w}\boldsymbol{a}_+\boldsymbol{a}_- - \boldsymbol{a}_+\boldsymbol{a}_-\boldsymbol{w})\\ &= -\boldsymbol{\varepsilon}[\boldsymbol{a}_+\boldsymbol{a}_-, \boldsymbol{w}] = -\boldsymbol{\varepsilon}[\boldsymbol{w} + R + \boldsymbol{\varepsilon}z, \boldsymbol{w}] = -\boldsymbol{\varepsilon}^2[z, \boldsymbol{w}] \end{aligned}$$

hence $(1+\boldsymbol{\varepsilon}^2)[z, \boldsymbol{w}] = 0$, so $[z, \boldsymbol{w}] = (1+\boldsymbol{\varepsilon}^2)^{-1}(1+\boldsymbol{\varepsilon}^2)[z, \boldsymbol{w}] = 0$. The unitarity of $\boldsymbol{u}$ (25) follow from the definition of $\boldsymbol{u}$ and the commutativity of z and $\boldsymbol{w}$.

From (17) and (27) we can have

$$\boldsymbol{z}\boldsymbol{a}_+ - \boldsymbol{a}_+\boldsymbol{z} = +\boldsymbol{\varepsilon}\boldsymbol{w}\boldsymbol{a}_+ + \boldsymbol{\varepsilon}\boldsymbol{a}_+\boldsymbol{w}, \qquad -\boldsymbol{\varepsilon}\boldsymbol{z}\boldsymbol{a}_+ - \boldsymbol{\varepsilon}\boldsymbol{a}_+\boldsymbol{z} = \boldsymbol{w}\boldsymbol{a}_+ - \boldsymbol{a}_+\boldsymbol{w},$$

solving these as simultaneous equations and using the tan half angle identity give the first of results of (28). The other identities in (28) follow similarly. The identities in (29) are then the complex version of (28).

The generators all $\mathcal{A}(R)$ are all of the form (30). Given $\boldsymbol{f}$ of this form, then $\boldsymbol{f}\boldsymbol{u}$, $\boldsymbol{f}\boldsymbol{\varepsilon}$ and $\boldsymbol{f}(1-\boldsymbol{\varepsilon}^2)^{-1}$ are all of the form (30). Also

$$\begin{aligned}\boldsymbol{f}\boldsymbol{a}_+ &= \sum_{r=0}^{N}\sum_{s=-N}^{N} \boldsymbol{a}_+^{r+1}\boldsymbol{u}^s\,\mathrm{e}^{\mathrm{i}s\boldsymbol{\alpha}}\xi_{r,s}(\boldsymbol{\varepsilon}) + \sum_{r=1}^{N}\sum_{s=-N}^{N} \boldsymbol{a}_-^{r-1} \\ &\quad\times\left(\frac{1}{2}(1+\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u} + \frac{1}{2}(1-\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}^{-1} + R\right)\boldsymbol{u}^s\,\mathrm{e}^{-\mathrm{i}s\boldsymbol{\alpha}}\xi_{-r,s}(\boldsymbol{\varepsilon}).\end{aligned}$$

So $\boldsymbol{f}\boldsymbol{a}_+$ is of the form (30), likewise for $\boldsymbol{f}\boldsymbol{a}_-$.

For uniqueness, from linearity, it is enough to show that if $\boldsymbol{f}$ is of the form (30) and $\boldsymbol{f}=0$ then $\xi_{r,s}=0$ for all r, s. By multiplying $\boldsymbol{f}$ with $(1+\boldsymbol{\varepsilon}^2)^M$ for sufficiently high M then we can assume $\xi_{r,s}$ are all polynomials. Let m be the largest degree of these polynomials. Now

$$0 = \pi(\boldsymbol{f}) = \sum_{r=0}^{N}\sum_{s=-N}^{N} \pi(\boldsymbol{a}_+)^r\pi(\boldsymbol{u})^s\xi_{r,s}(0) + \sum_{r=1}^{N}\sum_{s=-N}^{N} \pi(\boldsymbol{a}_-)^r\pi(\boldsymbol{u})^s\xi_{-r,s}(0),$$

so by looking at the coordinate system on $\mathcal{M}(R)$ this implies $\xi_{r,s}(0)=0$ for all r, s. Thus $\xi_{r,s}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}\hat{\xi}_{r,s}(\boldsymbol{\varepsilon})$ where $\hat{\xi}_{r,s}(\boldsymbol{\varepsilon})$ are polynomials of degree $\leq m-1$. Continuing this gives $\xi_{r,s}=0$. □

Lemma 2. *The relationships (19)–(29) define an associative algebra.*

Proof. Let $\mathcal{S}$ be set of all expressions of the form (30). We define a product $\cdot$ on $\mathcal{S}$ using the above definitions.

We wish to show that $\boldsymbol{f}_1\cdot(\boldsymbol{f}_2\cdot\boldsymbol{f}_3) = (\boldsymbol{f}_1\cdot\boldsymbol{f}_2)\cdot\boldsymbol{f}_3$ for $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3 \in \mathcal{S}$. Where the inner bracket must be written in the form (30) first. It is sufficient to show that $\boldsymbol{f}_1\cdot(\boldsymbol{f}_2\cdot\boldsymbol{f}_3) = (\boldsymbol{f}_1\cdot\boldsymbol{f}_2)\cdot\boldsymbol{f}_3$ where $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$ are from the set $\{\boldsymbol{u}, \boldsymbol{a}_+, \boldsymbol{a}_-, \boldsymbol{\varepsilon}, (1+\boldsymbol{\varepsilon}^2)^{-1}\}$. This is because any expression can be constructed from these five elements. The element $\boldsymbol{u}^\dagger = 2\boldsymbol{w} - \boldsymbol{u}$ with $\boldsymbol{w}$ given by (19). If $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ or $\boldsymbol{f}_3$ are either $\boldsymbol{\varepsilon}$ or $(1+\boldsymbol{\varepsilon}^2)^{-1}$ then the association relation holds since $\boldsymbol{\varepsilon}$ commutes with all the generators.

The remaining 27 relationships must be proved in turn, the interesting ones are

$$\begin{aligned}(\boldsymbol{u}\cdot\boldsymbol{a}_+)\cdot\boldsymbol{a}_- &= \boldsymbol{a}_+\boldsymbol{u}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}}\cdot\boldsymbol{a}_- = (\tfrac{1}{2}(1-\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u} + \tfrac{1}{2}(1+\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}^{-1} + R)\boldsymbol{u} \\ &= \boldsymbol{u}(\tfrac{1}{2}(1-\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u} + \tfrac{1}{2}(1+\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}^{-1} + R) = \boldsymbol{u}\cdot(\boldsymbol{a}_+\cdot\boldsymbol{a}_-)\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{a}_+\cdot\boldsymbol{a}_-)\cdot\boldsymbol{a}_+ &= (\tfrac{1}{2}(1-\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u} + \tfrac{1}{2}(1+\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}^{-1} + R)\cdot\boldsymbol{a}_+ \\ &= \boldsymbol{a}_+(\tfrac{1}{2}(1-\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}} + \tfrac{1}{2}(1+\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}^{-1}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}} + R) \\ &= \boldsymbol{a}_+(\tfrac{1}{2}(1+\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u} + \tfrac{1}{2}(1-\mathrm{i}\boldsymbol{\varepsilon})\boldsymbol{u}^{-1} + R) = \boldsymbol{a}_+\cdot(\boldsymbol{a}_-\cdot\boldsymbol{a}_+).\end{aligned}$$ □

3. Finite dimensional representations of $\mathcal{A}(R)$

As mentioned in Section 1, we wish to find irreducible representations over a finite Hilbert space $\mathcal{H}$ of $\mathcal{A}(R)$, i.e. $\Psi\colon \mathcal{A}(R) \mapsto \mathcal{L}(\mathcal{H},\mathcal{H}) \cong M_n(\mathbb{C})$. Such that $\Psi(\boldsymbol{\varepsilon}) = \varepsilon\mathbb{1}$ where $\varepsilon \in \mathbb{R}_+$ and such that Ψ preserves the adjoint operator so that $\Psi(\boldsymbol{f}^\dagger) = \Psi(\boldsymbol{f})^\dagger$ where the dagger on the right-hand side is the Hermitian conjugate. We note that since $\Psi(\boldsymbol{u})^\dagger\Psi(\boldsymbol{u}) = \mathbb{1}$ we can diagonalise $\Psi(\boldsymbol{u})$ and the eigenspaces of $\Psi(\boldsymbol{u})$ are orthogonal. We shall further assume that $\Psi(\boldsymbol{u})$ has no multiple eigenvalues, so the eigenspaces of $\Psi(\boldsymbol{u})$ are all one dimensional. This significantly simplifies the types of representations.

Theorem 3. *Let $\Psi\colon \mathcal{A}(R) \mapsto \mathcal{L}(\mathcal{H},\mathcal{H}) \cong M_n(\mathbb{C})$ be an irreducible adjoint preserving n dimensional representation, such that $\Psi(\boldsymbol{\varepsilon}) = \varepsilon\mathbb{1}$, $\varepsilon \in \mathbb{R}_+$ and $\Psi(\boldsymbol{u})$ has no multiple eigenvalues. Let α be given by $\tan((1/2)\alpha) = \varepsilon$, $0 < \alpha < \pi/2$. Then there exists $\beta \in \mathbb{R}$, such that we can label the orthonormal bases for $\mathcal{H}$ which are the eigenspaces $\Psi(\boldsymbol{u})$*

$$|\beta\rangle, |\beta+\alpha\rangle, |\beta+2\alpha\rangle, \dots, |\beta+(n-1)\alpha\rangle, \tag{31}$$

where

$$\boldsymbol{u}|\beta+m\alpha\rangle = \mathrm{e}^{\mathrm{i}(\beta+m\alpha)}|\beta+m\alpha\rangle, \quad m = 0,\dots,n-1. \tag{32}$$

There also exists a set of complex constants $C_{\beta+m\alpha} \in \mathbb{C}$ for $m = 0,\dots,n$ satisfying

$$|C_{\beta+m\alpha}|^2 = \sec(\tfrac{1}{2}\alpha)\cos(\beta - \tfrac{1}{2}\alpha + m\alpha) + R \tag{33}$$

and $C_{\beta+m\alpha} \neq 0$ for $m = 1,\dots,n-1$ so that

$$\begin{aligned} \boldsymbol{a}_-|\beta+m\alpha\rangle &= C_{\beta+m\alpha}|\beta+(m-1)\alpha\rangle, \quad m = 1,\dots,n-1,\\ \boldsymbol{a}_+|\beta+m\alpha\rangle &= \overline{C_{\beta+(m+1)\alpha}}|\beta+(m+1)\alpha\rangle, \quad m = 0,\dots,n-2. \end{aligned} \tag{34}$$

The action of $\boldsymbol{a}_-$ and $\boldsymbol{a}_+$ on the first and last vectors of (31), respectively, are given by either

$$\boldsymbol{a}_-|\beta\rangle = 0, \qquad \boldsymbol{a}_+|\beta+(n-1)\alpha\rangle = 0 \tag{35}$$

or

$$\boldsymbol{a}_-|\beta\rangle = C_\beta|\beta+(n-1)\alpha\rangle, \qquad \boldsymbol{a}_+|\beta+(n-1)\alpha\rangle = \overline{C_\beta}|\beta\rangle. \tag{36}$$

In the first case $C_\beta = C_{\beta+n\alpha} = 0$ satisfies (33). In the second case $C_\beta = C_{\beta+n\alpha} \neq 0$ satisfies (33) and

$$n\alpha = 2\pi k, \tag{37}$$

where $k \in \mathbb{N}$, $1 \leq k < n/2$ and n and k are relatively prime.

Proof. Let $|\theta\rangle$ be a normalised eigenvector of $\Psi(\boldsymbol{u})$ with eigenvalue λ then $|\lambda|^2 = \langle\theta|\boldsymbol{u}^\dagger\boldsymbol{u}|\theta\rangle = 1$ hence we can set $\lambda = \mathrm{e}^{\mathrm{i}\theta}$. From (22) we have $\boldsymbol{w}|\theta\rangle = \cos(\theta)|\theta\rangle$ and $\boldsymbol{z}|\theta\rangle = \sin(\theta)|\theta\rangle$. Let $N\colon \mathbb{R} \mapsto \mathbb{R}$ be given by $N(\theta) = \sec((1/2)\alpha)\cos(\theta - (1/2)\alpha) + R$. From (26) we have

$$\begin{aligned}\|\boldsymbol{a}_-|\theta\rangle\|^2 &= \langle\theta|\boldsymbol{a}_+\boldsymbol{a}_-|\theta\rangle = \langle\theta|(\boldsymbol{w}+\tan(\tfrac{1}{2}\boldsymbol{\alpha})\boldsymbol{z}+R)|\theta\rangle\\ &= \cos(\theta)+\tan(\tfrac{1}{2}\alpha)\sin(\theta)+R = N(\theta),\\ \|\boldsymbol{a}_+|\theta\rangle\|^2 &= \langle\theta|\boldsymbol{a}_-\boldsymbol{a}_+|\theta\rangle = \langle\theta|(\boldsymbol{w}-\tan(\tfrac{1}{2}\boldsymbol{\alpha})\boldsymbol{z}+R)|\theta\rangle\\ &= \cos(\theta)-\tan(\tfrac{1}{2}\alpha)\sin(\theta)+R = N(\theta+\alpha).\end{aligned}$$

Thus if $|\theta\rangle$ is a eigenvector then $N(\theta) \geq 0$ and $N(\theta+\alpha) \geq 0$. Furthermore $N(\theta) = 0 \Leftrightarrow \boldsymbol{a}_-|\theta\rangle = 0$ and $N(\theta+\alpha) = 0 \Leftrightarrow \boldsymbol{a}_+|\theta\rangle = 0$.

From (29) we have

$$\boldsymbol{u}\boldsymbol{a}_-|\theta\rangle = \boldsymbol{a}_-\boldsymbol{u}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}}|\theta\rangle = \boldsymbol{a}_-\,\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{e}^{\mathrm{i}\theta}|\theta\rangle = \mathrm{e}^{\mathrm{i}(\theta-\alpha)}\boldsymbol{a}_-|\theta\rangle.$$

Hence if $N(\theta) \neq 0$ then there must exist another normalised eigenvector $|\theta'\rangle$ with eigenvalue $\mathrm{e}^{\mathrm{i}(\theta-\alpha)}$. If we let $\boldsymbol{a}_-|\theta\rangle = C_\theta|\theta'\rangle$ then $|C_\theta|^2 = N(\theta)$. Thus C_θ is determined from $N(\theta)$ up to a phase.

Similarly $\boldsymbol{u}\boldsymbol{a}_+|\theta'\rangle = \mathrm{e}^{\mathrm{i}(\theta'+\alpha)}\boldsymbol{a}_+|\theta'\rangle = \mathrm{e}^{\mathrm{i}\theta}\boldsymbol{a}_+|\theta'\rangle$. Since the eigenspaces of $\Psi(\boldsymbol{u})$ are all one dimensional then $\boldsymbol{a}_+|\theta'\rangle$ is parallel to $|\theta\rangle$. Thus setting $\boldsymbol{a}_+|\theta'\rangle = D_\theta|\theta\rangle$ then

$$D_\theta = \langle\theta'|\boldsymbol{a}_+|\theta\rangle = \overline{\langle\theta|\boldsymbol{a}_-|\theta'\rangle} = \overline{C_{\theta'}}$$

hence $\boldsymbol{a}_+|\theta'\rangle = \overline{C_{\theta'}}|\theta\rangle$.

Claim. We now make the following claim. Let $r \leq n$ and there exists a set of independent normalised eigenvectors $\{|\beta+m\alpha\rangle | m = 0,\dots,r-1\}$ such that $N(\beta+m\alpha) > 0$ for $m = 1,\dots,r-1$. If either

$$N(\beta) > 0 \ \text{ and } \ \boldsymbol{a}_-|\beta\rangle = C_\beta|\beta+(r-1)\alpha\rangle \ \text{ for some choice of } \ C_\beta$$

or

$$N(\beta) = 0, \ \text{ and } \ N(\beta+r\alpha) = 0$$

then $r = n$.

Proof of claim. Let $V_r = \mathrm{span}\{|\beta+m\alpha\rangle | m = 0,\dots,r-1\}$. Since $N(\beta+m\alpha) > 0$ for $m = 1,\dots,n-1$ then we can define $C_{\beta+m\alpha} \neq 0$ for $m = 1,\dots,n-1$ so that $\boldsymbol{a}_-|\beta+m\alpha\rangle = C_{\beta+m\alpha}|\beta+(m-1)\alpha\rangle$ for $m = 1,\dots,r-1$ and $\boldsymbol{a}_+|\beta+m\alpha\rangle = \overline{C_{\beta+(m+1)\alpha}}|\beta+(m+1)\alpha\rangle$ for $m = 0,\dots,n-2$.

For the first option $\boldsymbol{a}_-|\beta\rangle = C_\beta|\beta+(r-1)\alpha\rangle$ for so $C_\beta \neq 0$, so from the argument above $\boldsymbol{a}_+|\beta+(r-1)\alpha\rangle = \overline{C_\beta}|\beta\rangle$.

For the second option $\boldsymbol{a}_-|\beta\rangle = 0$ and $\boldsymbol{a}_+|\beta+(r-1)\alpha\rangle = 0$.

Hence, for both cases, the action of $\boldsymbol{u}, \boldsymbol{a}_+, \boldsymbol{a}_-$ on V_r remains in V_r. Thus if $r < n$ then it is obvious that Ψ can be reduced to V_r. This contradicts the irreducibility of Ψ or the dimension $\mathcal{H}$ is n. Hence $r = n$. □

Continuation of proof. Since $N(\theta) \geq 0$ for all eigenvectors $|\theta\rangle$ we have two possibilities. Either $N(\theta) > 0$ for eigenvectors $|\theta\rangle$ or there exists a $|\beta\rangle$ such that $N(\beta) = 0$.

Taking the first case, that $N(\theta)| > 0$ all eigenvectors $|\theta\rangle$. Choose β so that $|\beta\rangle$ is any normalised eigenvector of $\Psi(\boldsymbol{u})$. We choose the phases of $C_{\beta+m\alpha}$, and define $|\beta+m\alpha\rangle = \overline{C_{\beta+m\alpha}}^{-1}\boldsymbol{a}_+|\beta+(m-1)\alpha\rangle$ for $m=1,\dots,n-1$.

From the claim above the $\{|\beta+m\alpha\rangle|m=0,\dots,n-1\}$ are independent. Since $N(\beta)>0$ then $\boldsymbol{a}_-|\beta\rangle$ must be parallel to one of the this set. But from the claim the only possibility is $\boldsymbol{a}_-|\beta\rangle = C_\beta|\beta+(n-1)\alpha\rangle$ for some choice of phase of C_β, i.e. (36).

Given condition (36) then

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\beta}\overline{C_\beta}|\beta\rangle &= \boldsymbol{u}\overline{C_\beta}|\beta\rangle = \boldsymbol{u}\boldsymbol{a}_+|\beta+(n-1)\alpha\rangle = \mathrm{e}^{\mathrm{i}\alpha}\boldsymbol{a}_+\boldsymbol{u}|\beta+(n-1)\alpha\rangle \\ &= \mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{e}^{\mathrm{i}(\beta+(n-1)\alpha)}\boldsymbol{a}_+|\beta+(n-1)\alpha\rangle = \mathrm{e}^{\mathrm{i}(\beta+n\alpha)}\overline{C_\beta}|\beta\rangle.\end{aligned}$$

Hence $\mathrm{e}^{\mathrm{i}n\alpha}=1$ so $n\alpha = 2\pi k$ for some integer k. Clearly $k\le 1$ and $k<n/2$ so that $0<\alpha<\pi$. Also k and n must be relatively prime so that $\Psi(\boldsymbol{u})$ has distinct eigenvalues, i.e. (37).

Now consider the second possibility. That there exist a $|\beta\rangle$ such that $N(\beta)=0$ and hence $\boldsymbol{a}_-|\beta\rangle = 0$. Again we choose the phases of $C_{\beta+m\alpha}$ and define $|\beta+m\alpha\rangle = \overline{C_{\beta+m\alpha}}^{-1}\boldsymbol{a}_+|\beta+(m-1)\alpha\rangle$ for $m=1,\dots,n-1$. These are all independent and none of the $C_{\beta+m\alpha}=0$ by the claim above.

If $\boldsymbol{a}_+|\beta+(n-1)\alpha\rangle \ne 0$ then by the claim above $\boldsymbol{a}_+|\beta+(n-1)\alpha\rangle = \overline{C_{\beta+n\alpha}}|\beta\rangle$. Hence

$$|C_{\beta+n\alpha}|^2 = \langle\beta+(n-1)\alpha|\boldsymbol{a}_-\boldsymbol{a}_+|\beta+(n-1)\alpha\rangle = \overline{C_{\beta+n\alpha}}\langle\beta+(n-1)\alpha|\boldsymbol{a}_-|\beta\rangle = 0$$

hence $C_{\beta+n\alpha}=0$ which contradicts $\boldsymbol{a}_+|\beta+(n-1)\alpha\rangle\ne 0$. Hence (35).

By analogy with the representations of the noncommutative sphere these we shall call representations which satisfy (35) S^2-*type representations* and we say a S^2-type representation is *minimal* if

$$n\alpha < 2\pi. \tag{38}$$

By analogy to the representations of the noncommutative torus we shall call representations which satisfy (36) T^2-*type representations* and we say a T^2-type representation is *minimal* if

$$n\alpha = 2\pi. \tag{39}$$

We now wish to find what values of (R,n,α,β) give rise to finite dimensional irreducible representations of $\mathcal{A}(R)$, with no multiple eigenvalues of $\Psi(\boldsymbol{u})$. A summary is given in Table 1.

3.1. S^2-type representations

Before we look at the different types of S^2-type representation and when they exist we shall give some basic facts about S^2-type representation.

Lemma 4. *If Ψ is an S^2 representation with (R,n,α,β) given in Theorem 3 then we can find a basis (31) such that $C_{\beta+m\alpha}\in\mathbb{R}_+$. Also there exist an equivalent representation $\tilde\Psi$ with that same values of (R,n,α) and with β replaced by $\tilde\beta$ where $-2\pi<\tilde\beta-(1/2)\alpha\le 0$.*

If Ψ_1 and Ψ_2 are S^2 representation with the same (R,n,α,β) then they are equivalent.

Table 1
Summary of possible representations for different R and ε

	Null	Point	Sphere	Variety	Sphere–Torus	Sphere–Torus bdd	Torus
R range $R_\varepsilon = (1+\varepsilon^2)^{1/2}$	$R \le -1$	$R = -1$	$-1 < R < 1$	$R = 1$	$1 < R < R_\varepsilon$	$R = R_\varepsilon$	$R > R_\varepsilon$
Topology of manifold $\mathcal{M}(R)$	Null[a]	Point	S^2 top[b]	2D variety[c]	–[d]	–[d]	T^2
Minimal S^2 representations	χ	χ	☺	☺	☺	χ	χ
Nonminimal S^2 representations	χ	χ	χ	χ	☺	☺	χ
Finite dim T^2 representations	χ	χ	χ	χ	☺[e]	☺[f]	☺
Semi ∞-dim T^2 representations	χ	χ	χ	χ	χ	☺	χ
∞-dim T^2 representations	χ	χ	χ	χ	χ	☺	☺

[a] No solution to equation.
[b] $\mathcal{M}(R)$ is convex if $R < 0$.
[c] The point $(x, y, z) = (0, 0, 0)$ not smooth.
[d] Since $(1+\varepsilon^2)^{1/2} = 1$ so no sphere–torus exists.
[e] β range is limited.
[f] β range excludes one point.

Proof. We can see that given α and β the basis (31) is only defined up to a phase, likewise $C_{\beta+m\alpha}$ is also only defined up to a phase. Thus given the set $\{\nu_m \in \mathbb{C} | \nu_m \overline{\nu_m} = 1, m = 0, \dots, n-1\}$, we can always make the following replacement:

$$|\beta + m\alpha\rangle \to |\beta + m\alpha\rangle' = \nu_m |\beta + m\alpha\rangle, \qquad C_{\beta+m\alpha} \to C'_{\beta+m\alpha} = \nu_{m-1}\overline{\nu_m} C_{\beta+m\alpha} \tag{40}$$

without changing Eqs. (32) and (34). Thus if we set $\nu_m = \nu_{m-1} C_{\beta+m\alpha}/|C_{\beta+m\alpha}|$ then all the $C'_{\beta+m\alpha}$ are real and positive.

Clearly replacing β with $\tilde{\beta} \to \beta + 2\pi k$ for $k \in \mathbb{N}$ does not change the representation. Therefore we can place $\tilde{\beta}$ in any 2π range. The one chosen makes the calculations below simpler.

If Ψ_1 and Ψ_2 are representation with the same (R, n, α, β) then, setting the $C_{\beta+m\alpha}$ to be real and positive, the bases (31) are the same (up to an overall choice phase), and the action (32), (34) on these basis elements are the same, therefore the representations are equivalent. □

For the rest of this section we assume we are given R and n, and we wish to find α and β so that $\Psi(R, n, \alpha, \beta)$ is S^2-type irreducible representation. Given α and β we write $\beta' = \beta - (1/2)\alpha$. For a general S^2 representation (33) and (35) imply

$$\cos(\beta') = \cos(\beta' + n\alpha) = -R\cos(\tfrac{1}{2}\alpha). \tag{41}$$

The first equality is solved by setting $\beta' + n\alpha = 2\pi k \pm \beta'$ for some $k \in \mathbb{N}$. This implies either

$$\alpha = \frac{2\pi k}{n} \tag{42}$$

or

$$\beta' = \pi k - \tfrac{1}{2}n\alpha. \tag{43}$$

We can now place some simple constraints on (R, n, α, β) such that $\Psi(R, n, \alpha, \beta)$ is an irreducible S^2 representation. From (41) we see that for Ψ to be an S^2 representation then

$$R \leq \sec(\tfrac{1}{2}\alpha) = (1 + \varepsilon^2)^{1/2} \tag{44}$$

since $\varepsilon = \tan((1/2)\alpha)$. By looking at (3) we see that $\Psi(R, n, \alpha, \beta)$ is an irreducible S^2 representation if and only if

$$\cos(\beta' + m\alpha) + R\cos(\tfrac{1}{2}\alpha) > 0, \quad m = 1, \ldots, n-1. \tag{45}$$

3.1.1. Minimal S^2 representation

The first result is on the existence and uniqueness of minimal S^2-type representations.

Lemma 5. *Given R and n there exists minimal S^2-type representation if and only if $-1 < R < \sec(\pi/n)$. This representation is unique and is given by*

$$\cos(\tfrac{1}{2}\alpha) + R\cos(\tfrac{1}{2}\alpha) = 0, \quad 0 < \alpha < 2\pi/n, \;\; \beta' = -\tfrac{1}{2}n\alpha. \tag{46}$$

Proof. If Ψ is minimal we must exclude (42). Since we choose β' so that $-2\pi < \beta' \leq 0$ from (43) we have $-2 < k - n\alpha/2\pi \leq 0$. Applying $0 < n\alpha/2\pi < 1$ we have

$$-2 < \frac{n\alpha}{2\pi} < k \leq 1 + \frac{n\alpha}{2\pi} < 1,$$

therefore $k = 0$ or -1.

We shall exclude the case $k = -1$. If $k = -1$ so that $\beta' = -\pi - (1/2)n\alpha$, therefore $-2\pi < \beta' < -\pi$. Now $\beta' + \alpha = -\pi - (1/2)(n-2)\alpha \leq -\pi$ since $n \geq 2$. Thus $-2\pi < \beta' < \beta' + \alpha \leq -\pi$. Since $\cos$ is strictly decreasing in the range $-2\pi \cdots -\pi$ we have $-1 \leq \cos(\beta' + \alpha) < \cos(\beta') < 1$. Thus $\cos(\beta' + \alpha) - \cos(\beta') = \cos(\beta' + \alpha) + R\cos((1/2)\alpha) < 0$ which contradicts (45).

Thus if Ψ is a minimal S^2 representation we have (46).

Now consider the function $\hat{R}(\alpha) = -\cos((1/2)n\alpha)/\cos((1/2)\alpha)$ for the range $0 < \alpha < 2\pi/n$. We observe that $\hat{R}(0) = -1$ and $\hat{R}(2\pi/n) = \sec(\pi/n)$ and

$$\begin{aligned}\hat{R}'(\alpha) &= (\cos(\tfrac{1}{2}\alpha))^{-2}(\tfrac{1}{2}n\alpha\cos(\tfrac{1}{2}\alpha)\sin(\tfrac{1}{2}n\alpha) - \tfrac{1}{2}\alpha\cos(\tfrac{1}{2}n\alpha)\sin(\tfrac{1}{2}\alpha)) \\ &= (\cos(\tfrac{1}{2}\alpha))^{-2}\tfrac{1}{2}\alpha((n-1)\cos(\tfrac{1}{2}\alpha)\sin(\tfrac{1}{2}n\alpha) + \sin(\tfrac{1}{2}(n-1)\alpha)) > 0.\end{aligned}$$

Thus $\hat{R}\colon \{\alpha | 0 < \alpha < 2\pi/n\} \mapsto \{R | -1 < R < \sec(\pi/n)\}$ is an invertible function so we can uniquely solve (46). This also implies that $-1 < R < \sec(\pi/n)$.

We now wish to show that an α, β' satisfying (46) is a representation. This simply requires showing (45) is satisfied. Consider separately the cases $m = 1, \ldots, \lfloor (1/2)n \rfloor$ and $m =$

$\lfloor(1/2)n\rfloor + 1, \ldots, n-1$. In the first case we have $-(1/2)n < m - (1/2)n \le 0$ this implies $-\pi \le -(1/2)n\alpha < (m-(1/2)n)\alpha \le 0$. Since cos is strictly increasing in this range we have $-1 \le \cos(-(1/2)n\alpha) < \cos(m\alpha - (1/2)n\alpha) \le 1$ and hence (45) is satisfied. Likewise if $m = \lfloor(1/2)n\rfloor + 1, \ldots, n-1$ then $0 \le m - (1/2)n < (1/2)n$ so $0 \le m\alpha - (1/2)n\alpha < (1/2)n\alpha \le \pi$. Since cos is strictly decreasing in this range we have $-1 \le \cos((1/2)n\alpha) < \cos(m\alpha - (1/2)n\alpha) \le 1$ and hence (45) is satisfied. □

The minimal S^2 representation may be said to be the closest to the representation of the true noncommutative sphere, since they exist for $-1 < R < 1$ when the commutative limit is a topologically the sphere, they are uniquely determined for each R by n, and $\varepsilon = \tan((1/2)\alpha) \to 0$ as $n \to \infty$.

As Lemma 5 shows there exists a minimal S^2 representation of dimension n for $-1 < R < \sec(\pi/n)$. The relationship between R, n and α is shown in Fig. 2a, $n = 2, 3, \ldots$. Given a permitted R and n we can draw a diagram to represent Ψ. This an open polygon inside a

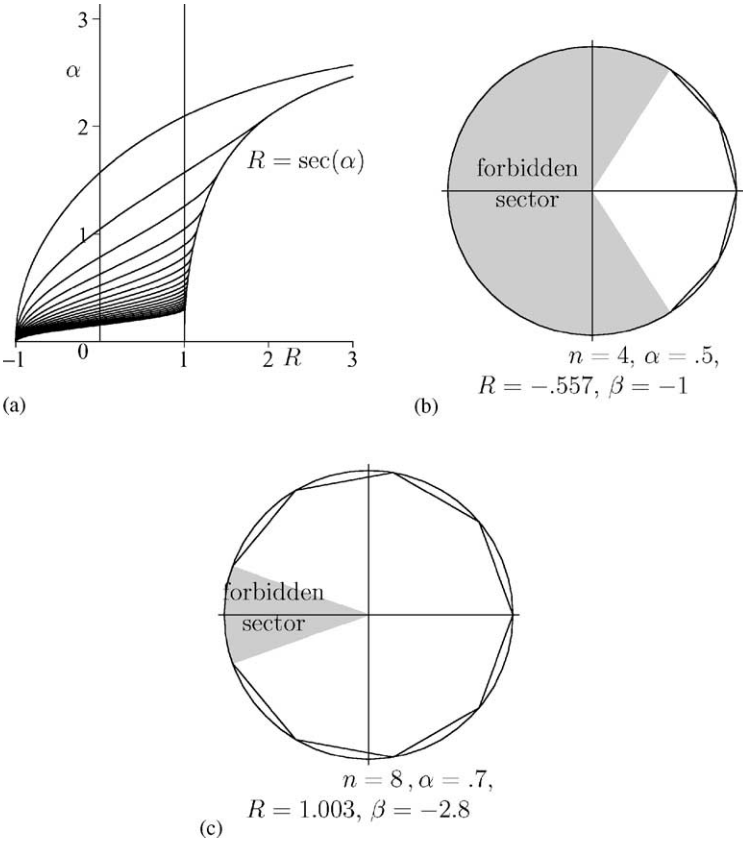

Fig. 2. The minimal S^2 representation. Part (a) gives the relationship between R, n and α for $n = 2, 3, \ldots$. Parts (b) and (c) represent different representations.

circle. The vertices are at the points $e^{i(\beta'+m\alpha)}$. Eq. (45) implies that there is a sector (called the “forbidden sector”) and all the vertices must lie to the right of this sector. The open polygon begins and ends on the edge of this sector. Fig. 2b and c represents representations for different n and R. Note for Fig. 2c, $R > 1$.

3.1.2. Nonminimal S^2 representation

The situation for nonminimal S^2 representation is more complicated. We have the following lemmas.

Lemma 6. *If Ψ is a nonminimal S^2-type representation then $R \geq 1$.*

Proof. If $|\theta\rangle$ is a basis vector then from (45) $\cos(\theta) \geq -R\cos((1/2)\alpha)$. Therefore if we set $\theta' = (\theta) \bmod 2\pi$ so that $0 \leq \theta' < 2\pi$, then θ' must lie either in the range $0 \leq \theta' \leq \pi - (1/2)\delta$ or $\pi + (1/2)\delta \leq \theta' < 2\pi$ where $\cos((1/2)\delta) = R\cos((1/2)\alpha)$ and $0 < \delta$. The range $\{\theta' | \pi - (1/2)\delta < \theta' < \pi + (1/2)\delta\}$ is called the forbidden sector.

Since Ψ is nonminimal there exists an m such that $0 \leq (\beta' + m\alpha) \bmod 2\pi \leq \pi - (1/2)\delta$ and $\pi + (1/2)\delta \leq (\beta' + (m+1)\alpha) \bmod 2\pi < 2\pi$. Thus $0 < \delta \leq \alpha < \pi$. Since cos is decreasing we have $0 < \cos((1/2)\alpha) \leq \cos((1/2)\delta) < 1$. Hence result. □

Lemma 7. *If $R \leq -1$ there exist no S^2-type representations.*

Proof. Follows from Lemmas 5 and 8. □

Lemma 8. *If $R \geq 1$ then there may exists many nonminimal S^2-type representation. Given such a $\Psi(R, n, \alpha, \beta)$ then $1 \leq R \leq \sec((1/2)\alpha)$ and $-3/2\pi < \beta' < -(1/2)\pi$ and also (R, n, α, β) obey either*

$$\cos\left(\frac{1}{2}n\alpha\right) = (-1)^{k+1}R\cos\left(\frac{1}{2}\alpha\right), \quad \frac{2\pi}{n} < \alpha < \frac{\pi}{2},$$
$$\beta' = \pi k - \tfrac{1}{2}n\alpha, \quad k = \left[\frac{n\alpha}{2\pi}\right] - 1 \tag{47}$$

or

$$\alpha = \frac{2k\pi}{n}, \quad 1 \leq k \leq \frac{(n-1)}{2}, \qquad \cos(\beta') = -R\cos\left(\frac{\pi k}{n}\right). \tag{48}$$

Proof. From (41) and $|\cos(\beta')| \leq 1$ we have $R \leq \sec((1/2)\alpha)$. Also since $R > 0$ and $\cos((1/2)\alpha) > 0$ then $\cos(\beta') < 0$ then $-(3/2)\pi < \beta' < -(1/2)\pi$.

From (41) we have either (42) or (43) is true. If (43) is true then from (41) the first equation in (47) is true. Also $\beta'/\pi = k - n\alpha/2\pi$ so $[\beta'/\pi] = -1 = k - [n\alpha/2\pi]$.

If (42) is satisfied, then from (41) Eq. (48) is satisfied. □

We note that the converse of Lemma 8 is not true. That is given (R, n, α, β) which satisfies either (47) or (48) there need not be a corresponding representation $\Psi(R, n, \alpha, \beta)$. This is due to the requirement that (45) is satisfied. For example given (R, n, α, β) which is a

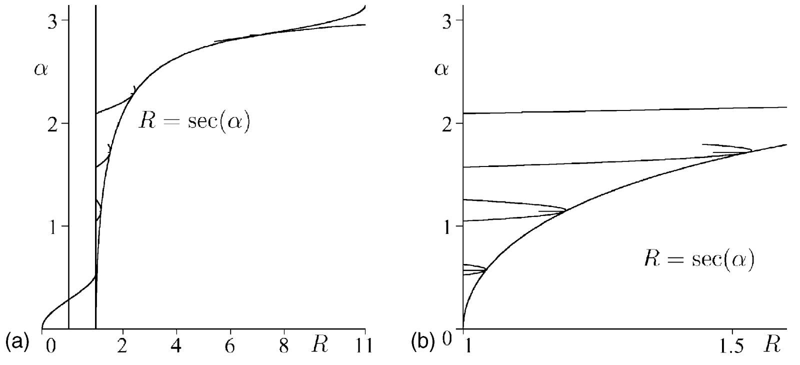

Fig. 3. Nonminimal S^2 representation, the relationship between R and α for $n = 11$.

solution to (47) such that n is even and k is old, then for $m = n/2$ we have

$$\cos(\beta' + m\alpha) + R\cos(\tfrac{1}{2}\alpha) = \cos(\pi k - \tfrac{1}{2}n\alpha + m\alpha) - \cos(\beta')$$
$$= \cos(\pi k) - \cos(\beta') = -1 - \cos(\beta') < 0,$$

hence (45) is not satisfied.

In general that exact set of (R, n, α, β) which have nonminimal S^2 representation is complicated. In Fig. 3 we see the relationship between R and α for $n = 11$. Fig. 3b is simply a smaller region of R. Fig. 4 gives four different representations with $n = 11$, the first two are acceptable since (45) is satisfied, whereas the second two are unacceptable since (45) is not satisfied. Fig. 4a and c corresponds to (47). Fig. 4b and d corresponds to (48).

3.2. T^2-type representations

Again before we look at the different types of T^2-type representation and when they exist, we shall give some basic facts about T^2-type representation. Recall that for a T^2-type representation $\alpha = 2\pi k/n$.

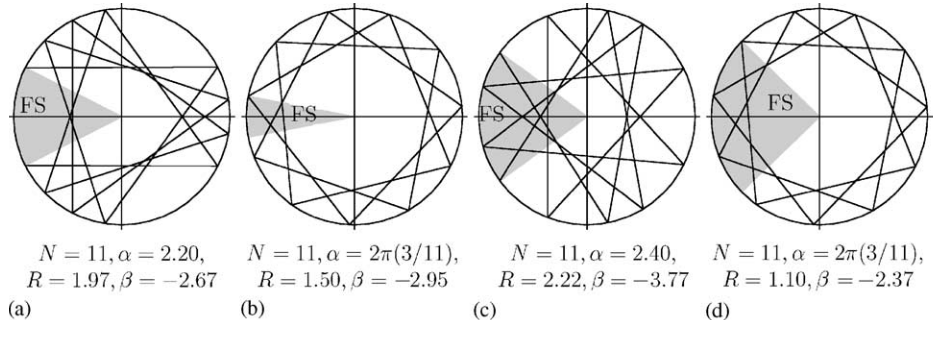

Fig. 4. Permissible and nonpermissible values of n, R, α, β (FS: forbidden sector).

We note take we can still make the replacement given by (40) for $m = 0, \dots, n-1$ with $\nu_{-1} = \nu_{n-1}$. However in this case the constant $C_{\text{prod}} = \prod_{m=0}^{n-1} C_{\beta+m\alpha}$ is unchanged under the replacement (40), i.e. $C_{\text{prod}} \to C'_{\text{prod}} = C_{\text{prod}}$. Thus there is a resulting phase $\nu = C_{\text{prod}}/|C_{\text{prod}}|$.

Lemma 9. *If Ψ_1 and Ψ_2 are T^2 representation with the same (R, n, k, β, ν) then they are equivalent.*

If Ψ is a T^2 representation we can find an equivalent representation $\tilde{\Psi}$ with the same (R, n, k, ν) and with β replaced by $\tilde{\beta}$ such that $\pi - 2\pi/n < \tilde{\beta} - (1/2)\alpha \le \pi$.

Proof. Since (R, n, k, ν) are the same for Ψ_1 and Ψ_2 then $\alpha = 2\pi k/n$ is the same for Ψ_1 and Ψ_2. By the same argument in Lemma 4 we can choose basis elements (31) such that $C_{\beta+m\alpha} \in \mathbb{R}_+$ for $m = 1, \dots, n-1$ and thus $C_\beta = \nu|C_\beta|$. Doing this for both Ψ_1 and Ψ_2 then the action (32), (34) on these basis elements are the same, therefore the representations are equivalent.

In the proof of Theorem 3 in the case of a T^2 representation when all $C_\theta \neq 0$ we choose β to be any value such that $e^{i\beta}$ was an eigenvalue of $\Psi(\boldsymbol{u})$. Since the n roots are equally spaced around the circle we can choose any arc of length $2\pi/n$ to place β in. The one chosen makes the calculations below simpler. □

For this section we assume we are given (R, n, k, ν), and we wish to find β so that $\Psi(R, n, k, \beta, \nu)$ is T^2-type irreducible representation. Again we define $\beta' = \beta - (1/2)\alpha$.

Lemma 10. *Given R, n and k such that $R < \cos(\pi/n)\sec(\pi k/n)$ there are no T^2 representation $\Psi(R, n, k, \beta, \nu)$.*

Given R, n and k such that $\cos(\pi/n)\sec(\pi k/n) < R \le \sec(\pi k/n)$ then there exist a one parameter set of T^2 representation $\Psi(R, n, k, \beta, \nu)$ with β in the range

$$\pi - \frac{2\pi}{n} + \frac{1}{2}\delta < \beta' < \pi - \frac{1}{2}\delta, \quad \text{where} \quad \cos\left(\frac{1}{2}\delta\right) = R\cos\left(\frac{\pi k}{n}\right). \tag{49}$$

Given R, n and k such that $R > \sec(\pi k/n)$ there exist a parameter set of irreducible T^2 representation $\Psi(R, n, k, \beta, \nu)$ with the full range of β, i.e. $\pi - 2\pi/n < \beta' \le \pi$.

Proof. By looking at (33) we see that $\Psi(R, n, k, \beta, \nu)$ is an irreducible T^2 representation if only if

$$\cos\left(\beta' + \frac{2\pi mk}{n}\right) + R\cos\left(\frac{\pi k}{n}\right) > 0, \quad m = 0, \dots, n-1, \tag{50}$$

hence if $R > \sec(\pi k/n)$, Eq. (50) is satisfied for all β', hence $\Psi(R, n, k, \beta, \nu)$ is a representation for all β.

For $R \le \sec(\pi k/n)$ then similar to Lemma 6 if we set $\theta' = (\beta' + 2\pi mk/n) \bmod 2\pi$ then θ' must lie either in the range $0 \le \theta' < \pi - (1/2)\delta$ or $\pi + (1/2)\delta < \theta' < 2\pi$ where δ is given in (49). Since the set $\{e^{i(\beta' + 2\pi mk/n)} | m = 0, \dots, n = 1\}$ are equally spaced then $\delta < 2\pi/n$. Hence $0 < (1/2)\delta < \pi/n \le (1/2)\alpha < (1/2)\pi$ and cos is strictly decreasing we have $0 < \cos(\pi/n) < \cos((1/2)\delta) < 1$. Since $\cos((1/2)\delta) = R\cos(\pi k/n)$ we have $R > \cos(\pi/n)\sec(\pi k/n)$.

Since $\beta' \leq \pi$ then $\beta' \leq \pi - (1/2)\delta$. Now there exists and m such that $\mathrm{e}^{\mathrm{i}(\beta'+2\pi mk/n)} = \mathrm{e}^{\mathrm{i}(\beta'+2\pi/n)}$, hence $(\beta' + 2\pi mk/n) \bmod 2\pi = \beta' + 2\pi/n$ hence $\beta' + 2\pi/n$ must lie either in the range $0 \leq \beta' + 2\pi/n < \pi - (1/2)\delta$ or the range $\pi + (1/2)\delta < \beta' + 2\pi/n < 2\pi$ but since $\beta' + 2\pi/n > \pi$ we have $\beta' > \pi + (1/2)\delta + 2\pi/n$. Thus for Ψ to be T^2 representation we must have β' given by (49).

If β' is given by (49) then all the θ' lie in the permitted regions hence (50) so it is a representation. □

We can see that Ψ is a minimal T^2 representation if $k = 1$. Thus we have the following corollary.

Corollary 11. *If $R \leq 1$ there are no T^2 representation.*

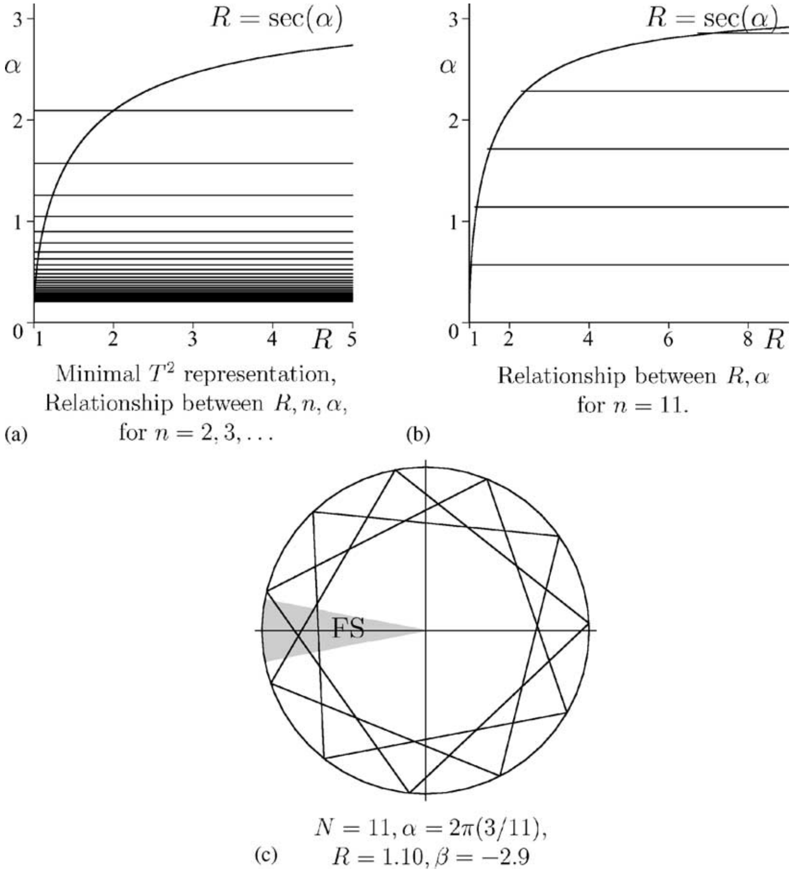

Fig. 5. Permissible region for T^2 representation and an example.

Given R and n such that $1 < R \leq \sec(\pi/n)$ *then there exist a one parameter set of irreducible* T^2 *minimal representation* $\Psi(R, n, k = 1, \beta, \nu)$ *with* β *given in* (49).

Given R and n such that $R > \sec(\pi/n)$ *there exist a one parameter set of irreducible* T^2 *representation* $\Psi(R, n, k = 1, \beta, \nu)$ *with the full range of* β, *i.e.* $\pi - 2\pi/n < \beta' \leq \pi$.

The range of possible (R, α) for minimal T^2 representation is simply $R > 1$ and $\alpha = 2\pi/n$. These are pictures in Fig. 5a. Fig. 5b gives the range of possible (R, α) for $n = 11$. Fig. 5c gives an example of a T^2 representation.

3.3. *Infinite dimensional* T^2 *representations*

Like the noncommutative torus $\mathcal{A}(R)$ also have infinite dimensional representations. We shall consider here only the infinite dimensional representations of the form (with $\mathcal{H}$ having the basis $\{|\beta + m\alpha\rangle | m \in \mathbb{Z}\}$)

$$\begin{aligned} &\boldsymbol{u}|\beta + m\alpha\rangle = \mathrm{e}^{\mathrm{i}(\beta+m\alpha)}|\beta + m\alpha\rangle, \qquad \boldsymbol{a}_-|\beta + m\alpha\rangle = C_{\beta+m\alpha}|\beta + (m-1)\alpha\rangle, \\ &\boldsymbol{a}_+|\beta + m\alpha\rangle = C_{\beta+(m+1)\alpha}|\beta + (m+1)\alpha\rangle, \end{aligned} \tag{51}$$

where

$$C_{\beta+m\alpha} = (\sec(\tfrac{1}{2}\alpha)\,\cos(\beta' + m\alpha) + R)^{1/2}, \quad \beta' = \beta - \tfrac{1}{2}\alpha \tag{52}$$

and where $\alpha \neq 2\pi n/k$ for any $n, k \in \mathbb{Z}$. The eigenvalues of $\Psi(\boldsymbol{u})$ are dense on the circle, so there exist an infinite dimensional representations if only if $R \geq \sec(\alpha)$.

If $R = \sec(\alpha)$ there also exist semi-infinite dimensional T^2 representation. These are given by $\beta' = -\pi$. Thus $\boldsymbol{a}_-|\beta\rangle = 0$ and $\boldsymbol{a}_+|\beta - \alpha\rangle = 0$ so we can reduce the Hilbert space to the subspaces $\mathrm{span}\{| -\pi + r\alpha\rangle, r \geq 0\}$ and $\mathrm{span}\{| -\pi + r\alpha\rangle, r \leq -1\}$.

4. Conclusion and discussion

Table 1 gives a list of all the possible representations. We can see that representations reflect the topology, but not completely.

There are loosely speaking four regions of R. If $R \leq -1$ then there is either no manifold $\mathcal{M}(R)$ or it is just a point. Consequently there are no representations either.

The next region $-1 < R < 1$ the algebra $\mathcal{A}(R)$ is closest to the noncommutative sphere. The commutative limit is the sphere, and there exist only minimal S^2 representations which are the closest to the representations of the noncommutative sphere, in that they are parameterised by n.

In the region $1 < R < (1+\varepsilon^2)^{1/2}$ the algebra $\mathcal{A}(R)$ is a new object which we may call the "sphere–torus". This is a purely noncommutative object since setting $\varepsilon = 0$ gives $1 < R < 1$ so there is no commutative analogue. In this region $\mathcal{A}(R)$ has the minimal S^2 representations, the nonminimal S^2 representations and the finite dimensional T^2 representations.

Only when $R > (1+\varepsilon^2)^{1/2}$ can we say that $\mathcal{A}(R)$ is a deformation of the torus. There are no S^2 representations and, as well as the finite dimensional T^2 representations, the infinite T^2 representations exist.

If any definition of genus is applied to noncommutative geometry it would be interesting to see what values it would attain for the sphere–torus.

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